Electrostatics: Image charges, Green's functions, and boundary value problems

Classical Mechanics and Electromagnetism for Accelerators

USPAS 2013

Gabriel Marcus

May 23, 2013

Introduction

The purpose of this handout is to illuminate special methods that can be used to solve for the electrostatic field, $\boldsymbol{E}(\boldsymbol{x})$, in the presence of arbitrary charge distributions and boundaries. The formalism will be presented generally while applications to accelerator systems will be emphasized for specific problems. It should be noted that no new material or methods are developed and any errors contained herein are strictly my own.

Electrostatics review

The behavior of the field $\boldsymbol{E}(\boldsymbol{x})$ in electrostatics can be completely described (up to the gradient of a scalar function that satisfies Laplace's equation) by specifying the curl and divergence everywhere in space. The two equations which govern this behavior are:

$$\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0 \tag{1}$$

and

$$\nabla \times \boldsymbol{E} = 0, \tag{2}$$

where $\rho = \rho(\boldsymbol{x})$ is the charge density and ϵ_0 is the permittivity of free space. Equation (1) is known as Gauss's Law while equation (2) follows directly from Coulomb's law and the knowledge that the curl of the gradient of a well behaved scalar function vanishes:

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{x}')\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,\mathrm{d}^3x' = -\frac{1}{4\pi\epsilon_0}\boldsymbol{\nabla}\int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'. \tag{3}$$

We can therefore define the scalar potential $\Phi(\boldsymbol{x})$ by

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi. \tag{4}$$

Plugging equation (4) into equation (1) yields the Poisson equation

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{5}$$

with the solution

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\ \mathrm{d}^3x'. \tag{6}$$

Finding the scalar potential is often much easier when solving electrostatic problems because it only requires finding one function of position rather than three, which is the case for the vector equation (3).

If electrostatic problems were always specified by continuous charge distributions in the absence of boundary surfaces, the solution to equation (5) would be the easiest to obtain. In reality, however, it is often the case that problems in electrostatics involve finite regions of space, with or without charge, that are bounded by surfaces with prescribed boundary conditions. Boundary surfaces can be introduced by consulting the divergence theorem for a well-behaved field, $\boldsymbol{A}$, in a volume, V, bounded by the surface S with a normal vector $\boldsymbol{n}$:

$$\int_V \boldsymbol{\nabla}\cdot\boldsymbol{A}\ \mathrm{d}^3x = \oint_S \boldsymbol{A}\cdot\boldsymbol{n}\ \mathrm{d}a. \tag{7}$$

Letting $\boldsymbol{A} = \varphi\boldsymbol{\nabla}\psi$ for the arbitrary scalar fields φ and ψ and using the relations

$$\begin{aligned} \boldsymbol{\nabla}\cdot(\varphi\boldsymbol{\nabla}\psi) &= \varphi\nabla^2\psi + \boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\psi, \\ \varphi\boldsymbol{\nabla}\psi\cdot\boldsymbol{n} &= \varphi\frac{\partial\psi}{\partial n}, \end{aligned} \tag{8}$$

we obtain Green's first identity

$$\int_V (\varphi\nabla^2\psi + \boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\psi)\ \mathrm{d}^3x = \oint_S \varphi\frac{\partial\psi}{\partial n}\ \mathrm{d}a \tag{9}$$

where $\partial\psi/\partial n$ is the outwardly directed normal derivative at the surface. If we interchange φ and ψ above and subtract the result from equation (9) we obtain Green's theorem:

$$\int_V (\varphi\nabla^2\psi - \psi\nabla^2\varphi)\ \mathrm{d}^3x = \oint_S \left[\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right]\ \mathrm{d}a. \tag{10}$$

The Poisson equation can be converted to an integral equation for the scalar potential by the appropriate choice of φ and ψ. Before proceeding, however, it is useful at this point to review the concept of *Green's functions* and their application to solving differential equations.

Green's functions (somewhat oversimplified)

Consider the following differential equation:

$$y'' + \omega^2 y = f(t), \tag{11}$$

where the prime($'$) indicates differentiation with respect to the independent variable t, ω is a constant, and $f(t)$ is an arbitrary forcing function. We can write the forcing function as a limiting case of a sequence of impulses:

$$f(t) = \int f(t')\delta(t'-t)\ \mathrm{d}t', \tag{12}$$

where $\delta(t'-t)$ is a dirac delta function. Suppose now that equation (11) was solved with $f(t) = \delta(t'-t)$, with the solution being $G(t,t')$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}G(t,t') + \omega^2 G(t,t') = \delta(t'-t). \tag{13}$$

Multiplying the above expression by $f(t')$ and integrating over t' one obtains

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int G(t,t')f(t')\ \mathrm{d}t' + \omega^2\int G(t,t')f(t')\ \mathrm{d}t' = \int \delta(t'-t)f(t')\ \mathrm{d}t'. \tag{14}$$

Upon comparison with equation (11) it is apparent that

$$y(t) = \int G(t,t')f(t')\ \mathrm{d}t'. \tag{15}$$

This rather brief (and somewhat incomplete) example serves to illustrate an important result. The Green function is the response of the system to a unit impulse at $t = t'$, and can be used to solve for the response of a system to an arbitrary forcing function.

Back to electrostatics

We are attempting to find solutions to Poisson's equation in the presence of boundary surfaces. To find this solution it is useful to find a Green's function that satisfies

$$\nabla'^2 G(\boldsymbol{x},\boldsymbol{x}') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{16}$$

with the appropriate boundary conditions. One solution that is immediately apparent upon comparison of equations (5), (6) and (16) is

$$G(\boldsymbol{x},\boldsymbol{x}') = \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} + F(\boldsymbol{x},\boldsymbol{x}'), \tag{17}$$

where $F(\boldsymbol{x},\boldsymbol{x}')$ (the significance of which will be discussed below) satisfies Laplace's equation in the volume, V, of interest:

$$\nabla'^2 F(\boldsymbol{x},\boldsymbol{x}') = 0. \tag{18}$$

Substituting $\varphi = \Phi$ and $\psi = G(\boldsymbol{x},\boldsymbol{x}')$ into Green's theorem we obtain the following:

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{x}')G(\boldsymbol{x},\boldsymbol{x}')\ \mathrm{d}^3x' + \frac{1}{4\pi}\oint_S \left[G(\boldsymbol{x},\boldsymbol{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\boldsymbol{x}')\frac{\partial G(\boldsymbol{x},\boldsymbol{x}')}{\partial n'}\right]\ \mathrm{d}a'. \tag{19}$$

The freedom afforded by $F(\boldsymbol{x}, \boldsymbol{x}')$ in equation (17) allows us to eliminate one term from the surface integral for the scalar potential in equation (19). For the remainder of this note, we will focus on *Dirichlet* boundary conditions where the Green's function vanishes on the surface and the scalar potential in the volume V is completely determined by the charge distribution (if any) and the value on the surface.

As a final remark before using the results obtained thus far for solving problems associated with accelerators, we note the physical meaning of $F(\boldsymbol{x}, \boldsymbol{x}')$. It is a solution to Laplace's equation in the volume of interest and therefore represents the potential due to a distribution of charge external to this volume. It is the potential of a distribution of charges, then, that satisfy Dirichlet (in our case) boundary conditions on the surface, S, when combined with the potential of a point charge located at $\boldsymbol{x}'$. Since the potential at $\boldsymbol{x}$ on the surface depends on the position of the point source (at $\boldsymbol{x}'$), the external charge distribution, $F(\boldsymbol{x}, \boldsymbol{x}')$, must also depend on $\boldsymbol{x}'$. In this sense, and for favorable geometries, we can often find the Green's function through the *Method of Images*.

Finding the Green's Function: Examples for Accelerators

Finding the correct Green's function for a particular geometry is sometimes easy and sometimes not. Three such methods that are often used include the method of images, orthonormal eigenfunction expansion, and the use of complex-variable techniques including conformal mapping. The method of images is probably the most common technique and as such will only be briefly (if at all) mentioned here. Conformal mapping is a bit beyond the scope of this class. Therefore, we will focus most of our attention on finding orthogonal functions that solve the underlying differential equation, and using them to build a Green's function.

The first example to be covered will be that of a long (in the longitudinal coordinate) beam traveling through a circular pipe, which will be presented in its entirety to illustrate the process in detail. We want to find the Green function for the interior Dirichlet problem for a cylinder of radius R. For a long beam the longitudinal coordinate is ignorable and we are left with a two dimensional problem. We seek solutions to

$$\nabla'^2 G(\boldsymbol{x}, \boldsymbol{x}') = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{20}$$

In cylindrical coordinates this becomes

$$\nabla^2 G(\rho, \phi; \rho'\phi') = \frac{-4\pi}{\rho}\delta(\rho - \rho')\delta(\phi - \phi'). \tag{21}$$

The ϕ delta function can be written in terms of an orthonormal function:

$$\delta(\phi - \phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}. \tag{22}$$

We therefore seek solutions to equation (21) that take the same form:

$$G(\rho, \phi; \rho'\phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\rho, \rho') \tag{23}$$

Expanding the Laplace operator in cylindrical coordinates we find:

$$\nabla^2 G(\rho, \phi; \rho'\phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dg_m}{d\rho} \right) - \frac{m^2}{\rho^2} g_m \right] e^{im(\phi-\phi')}. \tag{24}$$

The radial Green function then satisfies the equation

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dg_m}{d\rho} \right) - \frac{m^2}{\rho^2} g_m = \frac{-4\pi}{\rho} \delta(\rho - \rho'). \tag{25}$$

Two remarks are in order. First, for the Green function to be single valued on the interval $[0, 2\pi]$, m must be a positive or negative integer or zero. Second, we note that when $\rho \neq \rho'$, $g_m(\rho, \rho')$ is a solution to homogeneous Laplace equation with a different linear combination of the solutions to equation (25) for $\rho < \rho'$ and for $\rho > \rho'$ with a discontinuity in the first derivative at $\rho = \rho'$ determined by the delta function (remember quantum mechanics and the Schrödinger equation with a delta function potential?).

We propose a power solution $g_m = \rho^n$ to the homogeneous equation and find upon substitution that $n = \pm m$ for $m \neq 0$. We therefore have the following solutions for the indicated values of m:

$$\begin{aligned} g_m(\rho, \rho') &= \begin{cases} A\rho^m + B\rho^{-m} & \text{for } \rho < \rho' \\ A'\rho^m + B'\rho^{-m} & \text{for } \rho > \rho' \end{cases} \text{ for } m \geq 1, \\ g_m(\rho, \rho') &= \begin{cases} a + b\ln(\rho) & \text{for } \rho < \rho' \\ a' + b'\ln(\rho) & \text{for } \rho > \rho' \end{cases} \text{ for } m = 0. \end{aligned} \tag{26}$$

$g_m(\rho, \rho')$ must also be finite as $\rho \to 0$. Therefore, $B = b = 0$:

$$\begin{aligned} g_m(\rho, \rho') &= \begin{cases} A\rho^m & \text{for } \rho < \rho' \\ A'\rho^m + B'\rho^{-m} & \text{for } \rho > \rho' \end{cases} \text{ for } m \geq 1, \\ g_m(\rho, \rho') &= \begin{cases} a & \text{for } \rho < \rho' \\ a' + b'\ln(\rho) & \text{for } \rho > \rho' \end{cases} \text{ for } m = 0. \end{aligned} \tag{27}$$

Another condition for our Dirichlet boundary values is that $g_m(\rho = R, \rho') = 0$:

$$\begin{aligned} g_m(\rho, \rho') &= \begin{cases} A\rho^m & \text{for } \rho < \rho' \\ B'\left(-\frac{\rho^m}{R^{2m}} + \rho^{-m}\right) & \text{for } \rho > \rho' \end{cases} \text{ for } m \geq 1, \\ g_m(\rho, \rho') &= \begin{cases} a & \text{for } \rho < \rho' \\ b'\ln\left(\frac{\rho}{R}\right) & \text{for } \rho > \rho' \end{cases} \text{ for } m = 0. \end{aligned} \tag{28}$$

The solutions for the different regions must also match at $\rho = \rho'$:

$$\begin{aligned} g_m(\rho,\rho') &= \left\{ \begin{array}{ll} B'\left(-\frac{\rho^m}{R^{2m}} + \frac{\rho^m}{\rho'^{2m}}\right) & \text{for } \rho < \rho' \\ B'\left(-\frac{\rho^m}{R^{2m}} + \rho^{-m}\right) & \text{for } \rho > \rho' \end{array} \right. \text{ for } m \geq 1, \\ g_m(\rho,\rho') &= \left\{ \begin{array}{ll} b'\ln\left(\frac{\rho'}{R}\right) & \text{for } \rho < \rho' \\ b'\ln\left(\frac{\rho}{R}\right) & \text{for } \rho > \rho' \end{array} \right. \text{ for } m = 0. \end{aligned} \tag{29}$$

Finally, there is a discontinuity in the first derivative that can be found from equation (25):

$$\left.\frac{\partial g_m}{\partial \rho}\right|_{\rho=\rho'^+} - \left.\frac{\partial g_m}{\partial \rho}\right|_{\rho=\rho'^-} = -\frac{4\pi}{\rho'}. \tag{30}$$

After eliminating all of the constants we are left with the following equations:

$$\begin{aligned} g_m(\rho,\rho') &= \left\{ \begin{array}{ll} \frac{2\pi}{m}\left[\left(\frac{\rho}{\rho'}\right)^m - \left(\frac{\rho\rho'}{R^2}\right)^m\right] & \text{for } \rho < \rho' \\ \frac{2\pi}{m}\left[\left(\frac{\rho'}{\rho}\right)^m - \left(\frac{\rho\rho'}{R^2}\right)^m\right] & \text{for } \rho > \rho' \end{array} \right\} = \frac{2\pi}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{\rho\rho'}{R^2}\right)^m\right] \text{ for } m \geq 1, \\ g_m(\rho,\rho') &= \left\{ \begin{array}{ll} -4\pi\ln\left(\frac{\rho'}{R}\right) & \text{for } \rho < \rho' \\ -4\pi\ln\left(\frac{\rho}{R}\right) & \text{for } \rho > \rho' \end{array} \right\} \text{for } = -4\pi\ln\left(\frac{\rho_>}{R}\right) \quad m = 0. \end{aligned} \tag{31}$$

Here, $\rho_<(\rho_>)$ is the smaller(larger) of ρ and ρ'. We find that the Green function for this particular geometry and for Dirichlet boundary condition is

$$G(\rho,\phi;\rho',\phi') = \ln\left(\frac{R^2}{\rho_>^2}\right) + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{|m|}\left[\left(\frac{\rho_<}{\rho_>}\right)^{|m|} - \left(\frac{\rho\rho'}{R^2}\right)^{|m|}\right] e^{im(\phi-\phi')}, \tag{32}$$

where care must be taken for the sign of m when it is explicitly negative in $g_m(\rho,\rho')$. We can rewrite $G(\rho,\phi;\rho',\phi')$ as:

$$\begin{aligned} G(\rho,\phi;\rho',\phi') &= \ln\left(\frac{R^2}{\rho_>^2}\right) + \sum_{m=-\infty}^{-1} -\frac{1}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^{-m} - \left(\frac{\rho\rho'}{R^2}\right)^{-m}\right] e^{im(\phi-\phi')} + \ldots \\ &\ldots + \sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^{m} - \left(\frac{\rho\rho'}{R^2}\right)^{m}\right] e^{im(\phi-\phi')} \\ &= \ln\left(\frac{R^2}{\rho_>^2}\right) + \sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^{m} - \left(\frac{\rho\rho'}{R^2}\right)^{m}\right] e^{-im(\phi-\phi')} + \ldots \\ &\ldots + \sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^{m} - \left(\frac{\rho\rho'}{R^2}\right)^{m}\right] e^{im(\phi-\phi')} \\ &= \ln\left(\frac{R^2}{\rho_>^2}\right) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^{m} - \left(\frac{\rho\rho'}{R^2}\right)^{m}\right] \cos[m(\phi-\phi')]. \end{aligned} \tag{33}$$

We can manipulate this further by noting that

$$\frac{1}{m}x^m = \int_0^x \xi^{m-1}\mathrm{d}\xi = \int_0^x \frac{1}{\xi}\xi^m \mathrm{d}\xi. \tag{34}$$

Therefore, we can rewrite the individual terms in the sum in equation (33) for $\xi = \rho_</\rho_>$ as

$$2\sum_{m=1}^{\infty}\frac{1}{m}\xi^m\cos[m(\phi-\phi')] = 2\int_0^{\frac{\rho_<}{\rho_>}}\frac{1}{\xi}\left(\sum_{m=1}^{\infty}\xi^m\cos[m(\phi-\phi')]\right)\mathrm{d}\xi \tag{35}$$

Noting that $|\xi| < 1$ we find

$$\begin{aligned}
\sum_{m=1}^{\infty}\xi^m\cos[m(\phi-\phi')] &= \frac{1}{2}\sum_{m=1}^{\infty}\xi^m\left(e^{im(\phi-\phi')} + e^{-im(\phi-\phi')}\right)\\
&= \frac{1}{2}\left(\frac{\xi e^{i(\phi-\phi')}}{1-\xi e^{i(\phi-\phi')}} + \frac{\xi e^{-i(\phi-\phi')}}{1-\xi e^{-i(\phi-\phi')}}\right)\\
&= \frac{1}{2}\left(\frac{\xi\left[e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}\right] - 2\xi^2}{1+\xi^2-\xi\left[e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}\right]}\right)\\
&= \frac{\xi\cos(\phi-\phi')-\xi^2}{1+\xi^2-2\xi\cos(\phi-\phi')}.
\end{aligned} \tag{36}$$

Substituting this result into equations (35) and (33) we find

$$\begin{aligned}
2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos[m(\phi-\phi')] &= 2\int_0^{\frac{\rho_<}{\rho_>}}\frac{1}{\xi}\left(\sum_{m=1}^{\infty}\xi^m\cos[m(\phi-\phi')]\right)\mathrm{d}\xi\\
&= 2\int_0^{\frac{\rho_<}{\rho_>}}\frac{1}{\xi}\frac{\xi\cos(\phi-\phi')-\xi^2}{1+\xi^2-2\xi\cos(\phi-\phi')}\mathrm{d}\xi\\
&= 2\int_0^{\frac{\rho_<}{\rho_>}}\frac{\cos(\phi-\phi')-\xi}{1+\xi^2-2\xi\cos(\phi-\phi')}\mathrm{d}\xi\\
&= \int_1^{1+\left(\frac{\rho_<}{\rho_>}\right)^2-2\left(\frac{\rho_<}{\rho_>}\right)\cos(\phi-\phi')}\frac{\mathrm{d}u}{u}\\
&= -\ln\left(1+\left(\frac{\rho_<}{\rho_>}\right)^2 - 2\left(\frac{\rho_<}{\rho_>}\right)\cos(\phi-\phi')\right)
\end{aligned} \tag{37}$$

Similarly, we find that

$$2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho\rho'}{R^2}\right)^m\cos[m(\phi-\phi')] = -\ln\left(1+\left(\frac{\rho\rho'}{R^2}\right)^2 - 2\left(\frac{\rho\rho'}{R^2}\right)\cos(\phi-\phi')\right) \tag{38}$$

Putting all of this together, we find for $G(\rho, \phi; \rho', \phi')$:

$$
\begin{aligned}
G(\rho, \phi; \rho', \phi') &= \ln\left(\frac{R^2}{\rho_>^2}\right) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{\rho\rho'}{R^2}\right)^m\right]\cos[m(\phi-\phi')] \\
&= \ln\left(\frac{R^2}{\rho_>^2}\right) - \ln\left(1 + \left(\frac{\rho_<}{\rho_>}\right)^2 - 2\left(\frac{\rho_<}{\rho_>}\right)\cos(\phi-\phi')\right) + \ldots \\
&\ldots + \ln\left(1 + \left(\frac{\rho\rho'}{R^2}\right)^2 - 2\left(\frac{\rho\rho'}{R^2}\right)\cos(\phi-\phi')\right) \\
&= \ln\left(\frac{R^2}{\rho_>^2}\right) + \ln\left(\frac{1 + \left(\frac{\rho\rho'}{R^2}\right)^2 - 2\left(\frac{\rho\rho'}{R^2}\right)\cos(\phi-\phi')}{1 + \left(\frac{\rho_<}{\rho_>}\right)^2 - 2\left(\frac{\rho_<}{\rho_>}\right)\cos(\phi-\phi')}\right) \\
&= \ln\left(\left(\frac{R^4}{R^2\rho_>^2}\right)\frac{1 + \left(\frac{\rho\rho'}{R^2}\right)^2 - 2\left(\frac{\rho\rho'}{R^2}\right)\cos(\phi-\phi')}{1 + \left(\frac{\rho_<}{\rho_>}\right)^2 - 2\left(\frac{\rho_<}{\rho_>}\right)\cos(\phi-\phi')}\right) \\
&= \ln\left(\frac{R^4 + \rho^2\rho'^2 - 2\rho\rho' R^2\cos(\phi-\phi')}{R^2(\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi')}\right)
\end{aligned}
\tag{39}
$$